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Hypotheses behind light polarization studies

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1. Introduction

1.1 Disclaimer

Firstly, it is important to point out that the main interest of this text is to explore the polarization of electromagnetic waves itself, not focusing necessarily on the properties of these waves or of the mediums through which the radiation is traveling.

Moreover, linearity will be a central pillar on which we will build all of our analysis. Having said that, we will be working inside the limits of linear optics, even when it will not be explicitly written. More than that, we will be considering the conditions for the existence of the linear maps that we will present through these notes, especially those that are associated with the so-called Jones matrices.

1.2 Linear medium

As mentioned in 1.1, linearity is an extremely important feature for the developments presented here. Having said that, let us start with the definition of a linear medium, considering first the case in which this last one is a dielectric and, then, generalizing it. Note that it is just a pedagogical choice to justify the physics behind the definition for the polarization of materials, since we will be more interested in considering the medium in terms of its polarization than if it is just a dielectric, a metal, or other possible cases. As a first step, let us explain what a medium is.

Definition 1.1 (Medium and material)**.** *A medium is a pair (M, Υ), such that $M \in \mathbb{R}^3$ is a non-empty connected subset of $\mathbb{R}^3$ and Υ is a set of functions that describe the essential electromagnetic properties of M. Some renowned elements that usually constitute Υ are the permittivity tensors, the susceptibility tensors, the permeability tensors, the conductivity tensors, and the magnetoelectric coupling tensors.*
We generally will consider M as a domain, i.e., an open subset, when we need to work with spatial derivatives, and as a closed set when we integrate some function defined in M.
Moreover, when of the functions of Υ has a physical origin which explicitly, in a microscopic way, the macroscopic electromagnetic properties of M, M is usually called a material. In practice, the terms "medium" and "material" will be used as synonymous in the course of these notes.

Consider M a finite or possibly infinite dielectric medium with intrinsic electric dipoles and under the action of an external electric field $\vec{E}(\vec{r},t) \in \mathbb{R}^3$, where $\vec{r} \in \mathbb{R}^3$ is the position vector with respect to some arbitrary framework and $t > 0$ is the temporal parameter. Suppose that the intensity of $\vec{E}(\vec{r},t)$ is not high enough to break the dielectric strength of M, i.e., to make it a conductor. By the Lorentz force, these dipoles will tend to align to the direction of $\vec{E}(\vec{r},t)$ in counterpart to the entropic thermal effects. In addition, some dipoles may be induced in M due to the separation of charge in the polar constituents of M.

Then, let us avoid investigating the result of this phenomenon only from a macroscopic point of view, but, at the same time, let us consider the influence of sufficiently many dipoles on the general behavior of the system. In other words, we will be working on the mesoscopic scale.

Definition 1.2 (Mesoscopic scale)**.** *A material system is treated on the mesoscopic scale when its treatment occurs in the overlap of the microscopic and macroscopic scales, allowing analyses of the system both as interacting particles and as a continuum. Then, a volume V on this scale is called a mesoscopic scale.*

Armed with this notion, assume that the electric field has non-spatial-dependent, i.e., $\vec{E}(\vec{r},t) = \vec{E}(t)$ (assumption that we will address better soon). Then, we can properly define the polarization of M. Note that we are not talking about the polarization of the incident wave, but about the effect of the last one on the medium. To deal with this ambiguous terminology in Physics, for example, Mr. Cloude uses in [5] the linguistic difference between American and British English to refer to the light polarization writing "*polarisation*", while he evokes the other phenomenon by the spelling "*polarization*". Here, we will not use the last one for both phenomena.

Definition 1.3 (Polarization of a medium)**.** *Let V be a mesoscopic volume of M and let $\{\vec{p}_i(t)\}$ be the dipole moments inside V for each $t > 0$. The polarization of M due to an electric field $\vec{E}(\vec{r},t)$ is*

$$\vec{P}(t) = \frac{1}{V}\sum_i \vec{p}_i(t).$$

For a large number of materials, $\vec{P}(t)$ is related to the electric field $\vec{E}(t)$ by

$$\vec{P}(t) = \varepsilon_0 \int_0^t \chi(t-\tau)\vec{E}(\tau)\,\mathrm{d}\tau, \tag{1}$$

where $\varepsilon_0 \in \mathbb{R}$ is the vacuum permittivity and $\chi : [0, +\infty) \to \mathbb{C}$ is called the 1-susceptibility tensor of the material (or simply the susceptibility tensor). It is important to point out that χ could be defined in the whole $\mathbb{R}$. Nevertheless, here, we are considering only causal mediums, i.e., $\chi(t) = 0$ for $t < 0$, which means that the medium does not respond to the applied field before its application. Due to this supposition, the domain of χ was restricted to the non-negative semiline. This supposition will hold throughout this text for all the physical quantities, so that all the present mediums will be causal. Additionally, note that the expression 1 is a convolution $\left(\chi \, *_t \vec{E}\right)$ of χ and $\vec{E}$ in the time, expressing the fact that the material can present aspects of memory. The integral, computed coordinate by coordinate, accumulates the past contributions of $\vec{E}(\tau \le t)$ for the polarization, weighting these by $\chi(t-\tau)$, which decays with $t - \tau$ and makes the contributions of more distant moments less significant. An important consideration about integrals that will run through all of these notes is that we will consider all functions in integrands to be integrable. Just for completeness, to separate cases like that, with memory, from the others, let us define what is a medium with instantaneous response.

Definition 1.4 (Medium with instantaneous response)**.** *Let M be a medium and consider a physical response $\vec{A} : \mathbb{R}^3 \times [0, +\infty) \to \mathbb{R}^3$ to a field $\vec{F} : \mathbb{R}^3 \times [0, +\infty) \to \mathbb{R}^3$, given by*

$$\vec{A}(\vec{r}, t) = \int_0^t K(\vec{r}, \tau)\vec{F}(\vec{r}, t - \tau)\, \mathrm{d}\tau,$$

where $K : \mathbb{R}^3 \times [0, +\infty) \to \mathbb{K}$, for $\mathbb{K}$ equals to $\mathbb{R}$ or $\mathbb{C}$. If

$$K(\vec{r}, \tau) = K(r)\delta(\tau),$$

where $\delta : [0, +\infty) \to \{0, 1\}$ is the Dirac delta function, we say that M responds instantaneously to $\vec{F}$. When it happens,

$$\vec{A}(\vec{r}, t) = K(\vec{r})\vec{F}(\vec{r}, t)$$

and, for each $\Delta > 0$, $\vec{A}(\vec{r}, t_0)$ doesn't influence $A(\vec{r}, t_0 + \Delta)$, for all $t_0 \geq 0$. When M responds instantaneously to every physical field F (especially the fields $\vec{E}$ and $\vec{H}$), we say that M is a perfect medium.

Having made this comment, in order to clarify the linearity in 1, let us consider the electric field $\hat{E}(\omega)$ to be monofrequencial and represented in the frequency domain, and which is related to $\vec{E}(t)$ by

$$\vec{E}(t) = \Re\left\{\hat{E}(\omega)e^{i\omega t}\right\}. \tag{2}$$

Basically, we are working with Fourier analysis, but it will become clear when we discuss the harmonic regime. Just to make the notation clearer, the "*hat*" on the physical quantities expresses that they are being considered in the frequency domain. Having explained this consideration, the equation 1 reduces to

$$\hat{P}(\omega) = \varepsilon_0 \hat{\chi}(\omega)\hat{E}(\omega), \tag{3}$$

where

$$\hat{\chi}(\omega) = \int_0^\infty \chi(t)e^{-i\omega t}\, \mathrm{d}t\,.$$

Hence, the connection between the displacement electric field $\hat{D}(\omega)$, which considers the effects of inner polarizations, may be given by

$$\hat{D}(\omega) = \varepsilon_0\hat{E}(\omega) + \hat{P}(\omega) = \varepsilon_0\left(1 + \hat{\chi}(\omega)\right)\hat{E}(\omega), \tag{4}$$

where is clear the linear dependence of $\hat{D}(\omega)$ in $\hat{E}(\omega)$, connected by the 1-permittivity tensor $\hat{\varepsilon} : [0, +\infty) \to \mathbb{C}$ given by $\hat{\varepsilon} = 1 + \hat{\chi}(\omega)$.

For completeness, it is interesting to say that, due to the almost alignment of the dipole moments (intrinsic and induced), an electric field majority within the material arises with the opposite orientation of $\vec{E}(t)$, reducing the intensity felt by an arbitrary non-null charge in M.

On the other hand, when we think about the metals, the analysis is a little more complex because of the presence of metallic bonds instead of the prevalence of ionic and covalent bonds. In order to illustrate it, consider M a metallic medium under the incidence of an electromagnetic wave with electric field $\hat{E}(\omega)$, which we represented in the frequency domain. Unlike the dielectric case, the answer to the field is not limited only to the subtle displacement of the bound charges and the alignment of the electric dipoles, and this is because a considerable amount of electrons are not strongly bound to some specific atomic nucleus. Then, under the influence of $\hat{E}(\omega)$, they can collectively move through M, producing a conduction current $\hat{J}(\omega)$ usually expressed by

$$\hat{J}(\omega) = \hat{\sigma}(\omega)\hat{E}(\omega), \tag{5}$$

where $\hat{\sigma} : [0, +\infty) \to \mathbb{C}$ is called the 1-conductivity tensor of the material (or simply conductive tensor). Even having a resulting conduction current, it is still possible to consider the linearity

$$\hat{D}(\omega) = \hat{\varepsilon}_{eff}(\omega)\hat{E}(\omega), \tag{6}$$

where $\hat{\varepsilon}_{eff} : [0, +\infty) \to \mathbb{C}$ is the effective permittivity tensor of the material, which considers $\hat{\varepsilon}, \hat{\sigma} : [0, +\infty) \to \mathbb{C}$ both the electric permittivity tensor and the conductivity tensor. In other words, given $\hat{\varepsilon}$ and $\hat{\sigma}$, we could define $\hat{\varepsilon}_{eff} : \mathbb{C}^2 \to \mathbb{C}$ such that we could write explicitly

$$\hat{D}(\omega) = \hat{\varepsilon}_{eff}\left(\hat{\varepsilon}(\omega), \hat{\sigma}(\omega)\right)\hat{E}(\omega).$$

Typically, evoking the boundary conditions for the Maxwell equations, $\hat{E}(\omega)$ penetrates little in M and decays exponentially in a phenomenon called *skin depth*. In the limit case of a perfect conductor, $\hat{D}(\omega), \hat{E}(\omega) \to 0$ in M.

Notice that the hypothesis of a simple linear relation between some physical quantities was highlighted during this short explanation. We did not deal with magnetic phenomena itself during these lucubrations, however, let us assume that the magnetizing field $\hat{H}(\omega)$ does not depend on $\vec{r}$ and is related to the magnetic field $\hat{B}(\omega)$ by

$$\hat{B}(\omega) = \hat{\mu}(\omega)\hat{H}(\omega), \tag{7}$$

where $\hat{\mu} : [0, +\infty) \to \mathbb{C}$ is called 1-permeability tensor of the material (or simply permeability tensor). With this last ingredient in hand, we can use 4, 6, 5, 7 to explain what a linear medium is.

Definition 1.5 (Linear medium)**.** *Let M be a medium and consider $\mathcal{E}$ and $\mathcal{H}$ the spaces of electric fields and magnetizing fields, respectively, in M. Analogously, let $\mathcal{D}, \mathcal{J}$ and $\mathcal{B}$ be the spaces of the displacement fields, conduction currents, and magnetic fields, in this order, in M. Define the constitutive map of M by*

$$\begin{aligned} \Phi_M : \mathcal{E} \times \mathcal{H} &\to \mathcal{D} \times \mathcal{J} \times \mathcal{B} \\ \left(\vec{E}, \vec{H}\right) &\to \left(\vec{D}, \vec{J}, \vec{B}\right), \end{aligned}$$

which associates the physical fields above in M. Consider the case in which this map is linear, i.e.,[1]

$$\Phi\left(\Lambda_1\vec{E}_1+\Lambda_2\vec{E}_2,\Lambda_1\vec{H}_1+\Lambda_2\vec{H}_2\right)=\Lambda_1\Phi\left(\vec{E}_1,\vec{H}_1\right)+\Lambda_2\Phi\left(\vec{E}_2,\vec{H}_2\right)$$

for all $\vec{E}_1,\vec{E}_2\in\mathcal{E},\vec{H}_1,\vec{H}_2\in\mathcal{H},\Lambda_1,\Lambda_2\in\mathrm{Mat}_{3\times3}(\mathbb{K})$*, for* $\mathbb{K}$ *being* $\mathbb{R}$ *or* $\mathbb{C}$*.*[2] *When it happens, we say that* M *is a linear medium.*

Observe that what we are just saying is that, given two electric fields $\hat{E}_1,\hat{E}_2$ and $\Lambda_1,\Lambda_2\in\mathrm{Mat}_{3\times3}(\mathbb{K})$, we are able to use the superposition principle and write

$$\vec{D}=\varepsilon\left(\Lambda_1\vec{E}_1\right)+\varepsilon\left(\Lambda_2\vec{E}_2\right)=\Lambda_1\vec{D}_1+\Lambda_2\vec{D}_2,$$

where ε is the already presented permittivity tensor of the material and $\vec{D}_i:=\varepsilon\vec{E}_i$, for $i=1,2$. We can apply the same idea for $\vec{J}$ and $\vec{B}$. This superposition principle has, as a consequence, that if these fields are a composition of fields with different frequencies, then each one evolves independently and one frequency is not coupled to another.

Additionally, the subtlety of the definition 1.5 is in the fact that, in realistic problems, the physical fields are connected by the Maxwell equations and constrained by the boundary conditions which emerge from these equations. Then, if we consider only Maxwell physical solutions, the spaces of all possible fields may not have a structure similar to a vector space and the superposition principle might not work anymore. Nonetheless, for linear materials, it is always true and the spaces $\mathcal{E}$, $\mathcal{H}$, $\mathcal{D}$, $\mathcal{J}$ and $\mathcal{B}$ present a natural structure very similar to that of a vector space.[3]

That being said, we can analyze some possible origins for the non-linearity of a medium and its consequences. For some materials under certain conditions, their polarization due to an electric field $\vec{E}(t)$ depends on this field and its powers. Then, in a more general way, we can write $\vec{P}(t)$ as a power series of $\vec{E}(t)$, such that the coefficients are generalizations of the 1-permittivity tensor. Observe that the nomenclature for ε carries the terminology "*tensor*" and it is because it is a covariant tensor. Basically, given a finite vector space V of dimension n, for $k\in\mathbb{Z}_{\geq0}$, a $(k,1)$-tensor ζ on V is a multilinear map of the format

$$\zeta:V^*\times\underbrace{V\times\cdots\times V}_{k\text{ repetitions}}\to\mathbb{R}.$$

When the first entrance does not exist, i.e., when ζ is defined only in a product of spaces V, we say that ζ is a covariant tensor of order k.

Furthermore, when we have a $(k_1,1)$-tensor ζ_1 and a $(k_2,1)$-tensor ζ_2, we can define a $(k_1+k_2,2)$-tensor given by

$$\zeta_1\otimes\zeta_2\left(\alpha^1,\alpha^2,v_1,\ldots,v_{k_1+k_2}\right)=\zeta_1\left(\alpha^1,v_1,\ldots,v_{k_1}\right)\zeta_2\left(\alpha^2,v_{k_1+1},\ldots,v_{k_1+k_2}\right),$$

1. To make the notation cleaner, let us not write explicitly the space-temporal dependence of the vector fields in this definiton.
2. Formally, we define the product space $\mathcal{V}:=\mathcal{E}\times\mathcal{H}$ and consider the linearity of Φ for all linear combinations of elements $(\vec{E},\vec{H})$ of $\mathcal{V}$, where the coefficients of this combination are within $\mathrm{Mat}_{3\times3}(\mathbb{K})$.
3. A vector space is defined over a field in the sense of algebra. The space of matrices $\mathrm{Mat}_{3\times3}(\mathbb{K})$, with its canonical operations, is not a field, since singular elements do not have an inverse, but forms a ring.

for all $\alpha^1, \alpha^2 \in V^*,\ v_i \in V,\ i = 1, \ldots, (k_1 + k_2)$. Note that the second entrance of the pair (k, l) in the definition spelling of a (k, l)-tensor represents the number of dual spaces V^* in the product of the domain of the tensor. Consequently, we could present the definition of (k, l)-tensor, for l arbitrary, instead of taking k equal to 1 or 2. In fact, it is true, but it would introduce a denser notation than necessary, since the permittivity ε is just a $(k, 1)$-tensor.[4]

Thus, consider $\{b_i : i = 1, \ldots, n\}$ a basis for V, of which the dual basis is given by

$$\left\{\hat{b}^j \in V^* : \hat{b}^j(b_i) = \delta_i^j \ \forall\ i, j = 1, \ldots, n\right\}.$$

Consequently, we can describe ζ using these bases as

$$\zeta = \sum_{j, i_1, \ldots, i_k}^{n} \zeta\left(\hat{b}^j, b_{i_1}, \ldots, b_{i_k}\right) b_j \otimes \hat{b}^{i_1} \otimes \cdots \otimes \hat{b}^{i_k},$$

where the tensor product of elements of $\{b_i\}$ and $\left\{\hat{b}^j\right\}$ is an induced basis in the space of the $(k, 1)$-tensor on V.

A well-known result from the tensor calculus guarantees that there is a basis-independent isomorphism between the vector space of the multilinear maps

$$\underbrace{V \times \cdots \times V}_{k \text{ repetitions}} \rightarrow V$$

and the space of all $(k, 1)$-tensors on V. So, when $k = 1$, the space of these tensors is isomorphic to the space $\mathrm{End}(V)$ of endomorphisms of V, i.e., linear maps from V to V. Based on that, we can observe that both $\varepsilon(t)$ and $\sigma(t)$, for each $t \geq 0$, are $(1, 1)$-tensors, since they receive a vector $\vec{E}(t)$ written in a basis and return another vector $\vec{P}(t)$, up to a scale factor. In addition, note that $\vec{E}(t)$, for any fixed time t, or more generally for any element v of a vector space V, is itself a $(0, 1)$-tensor on V^* and it is due to the natural isomorphism between V and $(V^*)^*$, so that $v(\alpha) := \alpha(v),\ \forall\ \alpha \in V^*$.

Besides that, given an arbitrary $(k+1, l+1)$-tensor ζ, written in terms of a basis as

$$\zeta = \sum_{\substack{j_1, \ldots, j_{l+1} \\ i_1, \ldots, i_{k+1}}}^{n} \zeta\left(\hat{b}^{j_1}, \ldots, \hat{b}^{j_{l+1}}, b_{i_1}, \ldots, b_{i_{k+1}}, b_{i_{k+1}}\right) b_{j_1} \otimes \cdots \otimes b_{j_{l+1}} \otimes \hat{b}^{i_1} \otimes \cdots \otimes \hat{b}^{i_{k+1}},$$

define

$$\zeta^{j_1, \ldots, j_{l+1}}_{i_1, \ldots, i_{k+1}} := \zeta\left(\hat{b}^{j_1}, \ldots, \hat{b}^{j_{l+1}}, b_{i_1}, \ldots, b_{i_{k+1}}\right).$$

Then, we define the contraction of ζ in (i_m, j_m) in terms of basis as

$$\zeta^{j_1, \ldots, j_l}_{i_1, \ldots, i_l} := \sum_{i_m = 1}^{n} \zeta^{j_1, \ldots, i_m, \ldots, j_l}_{i_1, \ldots, i_m, \ldots, i_l},$$

4. To address the tensor theory in a more general way, see [6].

for $1 \leq m \leq \min\{k, l\}$. Note that it looks to be dependent on the basis, but it is a false impression. To see it, remember that the space of the $(1,1)$-tensors on V is canonically isomorphic to $\mathrm{End}(V)$. Thus, fixing all the entrances of ζ, except the mth covariant and contravariant, we have the endomorphism

$$\zeta(\alpha^1, \ldots, \bullet, \alpha^{m+1}, \ldots, \alpha^l, x_1, \ldots, \bullet, v_{m+1}, \ldots, v_k) \in \mathrm{End}(V).$$

As an endomorphism is invariant by changing the basis, the contraction is well-defined. For the sake of notation, when ζ is a $(k+1, 1)$-tensor and a sequence of contractions is applied to generate a $(1, 0)$-tensor, i.e., a vector, we use the notation $\zeta : (v_1, \cdots, v_{n+1}) \in V$. Explicitly, we can write

$$\zeta : (v_1, \cdots, v_{n+1}) := \sum_{j=1}^{n} \zeta^{j_2, \ldots, j_{k+1}} v_{j_2} \ldots v_{j_{k+1}}.$$

Despite that, returning to the supposition that $\hat{P}(\omega)$ does not depend linearly on $\hat{E}(\omega)$, we can write 3 in a more general way, using the tensor language, as

$$\hat{P}(\omega) = \hat{P}_0 + \varepsilon_0 \sum_{k \in \mathbb{N}} \hat{\chi}^{(k)}(\omega) : \left(\bigotimes_k \hat{E}(\omega) \right), \qquad \bigotimes_k \hat{E}(\omega) := \underbrace{\hat{E}(\omega) \otimes \cdots \otimes \hat{E}(\omega)}_{k \text{ repetitions}}, \tag{8}$$

where $\hat{P}_0 \in \mathbb{R}^3$, and where $\hat{\chi}^{(k)}(\omega)$ is a $(k, 1)$-covariant tensor, called the k-susceptibility tensor. As the second coordinate of $(k, 1)$ is fixed, let us say that $\hat{\chi}^{(k)}(\omega)$ is a k-covariant tensor (although it is, in fact, what we call a mixed tensor). When the material has no intrinsic polarization, $\hat{P}_0 = 0$, being this condition that we will suppose in these notes. Usually, only the first values of k effectively influence the physics observed. For this reason, let us make explicit the first terms of 9:

$$\hat{\chi}^{(1)}(\omega) : \left(\hat{E}(\omega) \right) = \sum_{j=1}^{3} \hat{\chi}^{(1)}_{ij}(\omega) \hat{E}_j(\omega),$$

$$\hat{\chi}^{(2)}(\omega) : \left(\hat{E}(\omega) \otimes \hat{E}(\omega) \right) = \sum_{j,k=1}^{3} \hat{\chi}^{(2)}_{ijk}(\omega) \hat{E}_j(\omega) \hat{E}_k(\omega),$$

$$\hat{\chi}^{(3)}(t) : \left(\hat{E}(\omega) \otimes \hat{E}(\omega) \otimes \hat{E}(\omega) \right) = \sum_{j,k,l=1}^{3} \hat{\chi}^{(3)}_{ijkl}(\omega) \hat{E}_j(\omega) \hat{E}_k(\omega) \hat{E}_l(\omega).$$

Notice that $\vec{P}(t)$ is still a vector, since we are considering contractions in 9, and the temporal-dependent can be explicitly reflected by the convolution $\left(\chi \ *_t \ \vec{E} \right)$, as

$$\vec{P}(\omega) = \vec{P}_0 + \varepsilon_0 \sum_{k \in \mathbb{N}} \chi^{(k)} *_t \left(\bigotimes_k \vec{E} \right), \tag{9}$$

where

$$\chi^{(k)} *_t \left(\bigotimes_k \vec{E} \right) := \int_0^t \cdots \int_0^t \chi^{(k)} \left(t - \tau_1, \ldots, t - \tau_k \right) : \left(\vec{E} \left(\tau_1 \right) \otimes \cdots \otimes \vec{E} \left(\tau_k \right) \right) \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_k \,.$$

Observe that we can generalize the definition 1.4 written $\chi^{(k)}$ in terms of a Dirac delta function. In addition, note that we are considering the case of a field monofrequential, i.e., with a unique ω. However, sometimes $\vec{E}(t)$ is a superposition of $n \in \mathbb{N}$ electric fields with different frequencies, so that we can write

$$\vec{E}(t) = \sum_{k=1}^{n} \frac{\hat{E}(\omega_k) e^{i\omega_n t} + \hat{E}^*(\omega_k) e^{-i\omega_n t}}{2}.$$

Consequently, when the exponential terms of $\vec{E}(t)$ are multiplied by each other, new components of frequency appear in the polarization, whose frequencies are the sum of the difference of the original ones. For instance, consider the terms $e^{i\omega_j t}$ and $e^{\pm i\omega_k t}$. Then, the product $e^{i\omega_j t}\, e^{\pm i\omega_k t} = e^{i(\omega_j \pm \omega_k)t}$ produces a new term with frequency $\omega_j \pm \omega_k$, which characterizes the phenomena of sum- or difference-frequency generation. But it is not the only combination of frequencies; there may be mixtures of waves generating combinations of three or more frequencies and, due to this, these compositions must be considered in 9, thus producing the expression

$$\hat{P}(\omega) = \varepsilon_0 \hat{P}_0(\omega) + \varepsilon_0 \sum_{k \in \mathbb{N}} \sum_{p \in S_k} \hat{\chi}^{(k)} \left(\omega_{p(1)}, \omega_{p(2)}, \ldots, \omega_{p(k)} \right) : \left(\hat{E} \left(\omega_{p(1)} \right) \otimes \cdots \otimes \hat{E} \left(\omega_{p(k)} \right) \right),$$

where $S_k := \{p : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}\}$ is the space of permutations of k elements.

Since it was explained, by 9, it is possible to see that the dependence of the polarization in the product $\otimes_k \vec{E}(t)$ removes the linearity that connects $\vec{D}(t)$ and $\vec{E}(t)$ in 6 for $k > 1$. Then, when the intensity of $\vec{E}(t)$ is strong enough to make the terms of order larger than one important in the polarization, the medium will respond non-linearly and the physical treatment of this system material-light falls within the scope of non-linear optics.

A famous example of a non-linear phenomenon in optics due to high-intensity incident light is the *Kerr effect*, which may be described briefly as the change of the refractive index of a material due to the presence of the intense electric field of this light. In order to understand it better, consider an electric field $\vec{E}(t)$ in a medium M and suppose that it can be written as

$$\vec{E}(t) = \frac{\hat{E}(\omega) e^{i\omega t} + \hat{E}(\omega)^* e^{-i\omega t}}{2},$$

for $\omega \in [0, +\infty)$ and where the vector notation was suppressed in $\hat{E}$, what we will do for the writing of the complex amplitude of fields throughout this text. Consider an arbitrary Cartesian framework. As there is no spatial dependence of $\vec{E}(t)$, the system formed by $\vec{E}(t)$ and M has inversion symmetry, i.e., the change $\vec{r} \to -\vec{r}$ in the position vector does not change either $\vec{E}(t)$, nor the field $\vec{D}(t)$. When the system has this kind of symmetry, $\chi^{(2k)} = 0$, $k \in \mathbb{N}$. It expresses the fact that the Kerr effect is a third-order effect on the electric field, while the permanence of only $\chi^{(1)}$ and $\chi^{(2)}$ determines what we call *electro-optics Pockels effect*. With that in mind, we can split $\vec{P}(t)$ into the linear $\vec{P}^L(t)$ and non-linear $\vec{P}^{NL}(t)$ parts: $\vec{P}(t) = \vec{P}^{L}(t) + \vec{P}^{NL}(t)$. In the high-intensity regime, the linear contributions are still present, but they are usually dominated by the non-linear term. For this reason, it is common in this scenario to refer to polarization as the non-linear part $P^{NL}(t)$ of $\vec{P}(t)$, a convention that we'll adopt here.

Having said that, in the context of the Kerr effect, the non-linear polarization is sensitive to interactions of electric fields oscillating at different frequencies. Note that all possible interactions between three electric fields with different frequencies, say $\omega_1, \omega_2, \omega_3 \in [0, +\infty)$, which can contribute to a polarization in a new frequency ω_4, may be expressed by the constraint

$$\omega_4 = \omega_1 + \omega_2 + \omega_3. \tag{10}$$

Consequently, assuming that $\vec{P}(t)$ follows the same format as $\vec{E}(t)$, i.e.,

$$\vec{P}(t) = \frac{\hat{P}(\omega)e^{i\omega t} + \hat{P}(\omega)^{*}e^{-i\omega t}}{2},$$

by 9, considering multiple frequencies connected by the 10, we have that the ith coordinate of $\hat{P}(\omega_4)$ is given by

$$\hat{P}_i\left(\omega_4\right) = \frac{1}{4}\varepsilon_0 \sum_{p\in S_3} \sum_{jkl=1}^{3} \chi^{(3)}_{ijkl}\left(\omega_{p(1)}, \omega_{p(2)}, \omega_{p(3)}\right) \hat{E}_j\left(\omega_{p(1)}\right) \hat{E}_k\left(\omega_{p(2)}\right) \hat{E}_l\left(\omega_{p(3)}\right). \tag{11}$$

Hence, when 10 is $\omega = 0 + 0 + \omega$, we obtain the so-called *electro-optical Kerr effect*, sometimes referred to as the DC Kerr effect, where two static fields and one oscillating interact in M resulting in a polarization at ω. To illustrate it, consider that we have only one strong static electric field, say along the direction y, and the other one can be taken as null. The presence of this intense field produces a difference between the refractive index $n_{\perp}(\omega)$ and $n_{/\!/}(\omega)$ measured along the direction y and x, respectively. Let us analyze how it is possible.

For some mediums[5] M, it is possible to reduce the tensorial expression 9 to a simple scalar equation

$$\hat{P}(\omega) = \varepsilon_0 \chi_{eff}(\omega)\hat{E}(\omega) + \mathcal{O}\left(\left\|\hat{E}(\omega)\right\|^4\right), \qquad \chi_{eff}(\omega) := \chi^{(1)}(\omega) + 3\chi^{(3)}(\omega)\left\|\hat{E}(\omega)\right\|^2,$$

or, conveniently,

$$\hat{P}(\omega) = \varepsilon_0\left[\chi^{(1)}(\omega) + \frac{3}{4}\chi^{(3)}(\omega)\hat{E}(\omega)\cdot\hat{E}(\omega)\right]\hat{E}(\omega)\cos(\omega t) + \varepsilon_0\frac{1}{4}\chi^{(3)}(\omega)\hat{E}(\omega)\cos(3\omega t). \tag{12}$$

Besides that, the refractive index of several materials obeys the relations

$$\begin{cases} n(\omega) = n_0(\omega) + n_2(\omega)\langle E^{\dagger}(\omega)\cdot E(\omega)\rangle, \\ n(\omega)^2 = 1 + \chi_{eff}(\omega), \end{cases}$$

5. Here, we will considering the hypothesis of isotropy and homogeneity for the medium, which will be defined soon.

where $\langle \cdot \rangle$ is the temporal average, n_0 is the refractive index in the presence of negligible fields,[6] $n_2(\omega)$ introduces the dependence between n and the intensity of the applied field. Then, a physical solution for this system of equations is

$$n_0(\omega) = \sqrt{1+\chi^{(1)}(\omega)}, \quad n_2 = \frac{3\chi^{(3)}(\omega)}{4n_0(\omega)}.$$

Observe that, when $\chi^{(3)}(\omega) \approx 0$, $n_2(\omega) \approx 0$ and the index $n(\omega)$ reduces to $n_0(\omega)$. Applying this process for each direction, we obtain $n_\perp(\omega)$ and $n_{/\!/}(\omega)$ from the expressions for each coordinate of the polarization, i.e., from[7]

$$\begin{aligned}\hat{P}_1(\omega) &= 3\varepsilon_0\chi^{(3)}_{1221}\,(0,0,\omega)\,E_2^2(0)\hat{E}_x(\omega),\\ \hat{P}_2(\omega) &= 3\varepsilon_0\chi^{(3)}_{2222}\,(0,0,\omega)\,E_2^2(0)\hat{E}_2(\omega).\end{aligned}$$

After that, let us consider the case $\omega_1 = \omega_2 - \omega_2 + \omega_1$, which concerns the so-called *optical Kerr effect*, also known as the AC Kerr effect. In this phenomenon, a wave of frequency ω_2 changes the refractive index of the medium to another wave of frequency ω_2, since the intensity of the first one is sufficiently greater than that of the second one. So, if we follow the steps presented for the electro-optical Kerr effect, we will observe that $n_2(\omega_1, \omega_2)$ will be a function of ω_1 and ω_2 due to a process called cross-phase modulation. As $n_2(\omega_1, \omega_2)$ will not be identically null, we can observe the change in the index experienced by the wave of frequency ω_1.

Accordingly, this is the main idea behind the Kerr effect.[8] Although the physical consequences of nonlinearity in the Kerr effect seem to be limited to changing the refractive index, let us observe carefully 12 and separate the expression into two terms:

$$\begin{aligned}P_\omega &:= \varepsilon_0\left[\chi^{(1)}(\omega) + \tfrac{3}{4}\chi^{(3)}(\omega)\hat{E}(\omega)\cdot\hat{E}(\omega)\right]\hat{E}(\omega)\cos(\omega t),\\ P_{3\omega} &:= \tfrac{\varepsilon_0}{4}\chi^{(3)}(\omega)\hat{E}(\omega)\cos(3\omega t).\end{aligned}$$

The term $P_{3\omega}$ is related to a process called third-harmonic generation, where three photons of frequency ω are annihilated to generate another photon with frequency 3ω. This kind of phenomenon is generalized for n photons, although often it is more difficult to generate a n-th harmonic as n increases. Consequently, the second harmonic is the first non-linear expression of this phenomenon. Notice that, when $\hat{E}(\omega)\cdot\hat{E}(\omega)$ is proportional to $\cos^2(\omega t)$, by the trigonometric identity

$$\cos^2(\omega t) = \frac{1+\cos(2\omega t)}{2},$$

the medium answers to this field by emitting light with double frequency 2ω, making red light become blue, for instance.

Therefore, we can observe that, for high intensity, the limits of linearity may be broken. To clarify it, we used the linear relation expressed in 6, however, we could also use 5 and

6. Note that it depends on $\omega \in [0, +\infty)$, even representing the refractive index in the absence of strong fields. It is because the realistic refractive indexes usually consider dependences of ε and μ on the frequency. In the literature, n_0 is called the linear refractive index.
7. As we will see in 1.5, which means that, in the presence of an intense electric field, this material expresses properties of birefringence.
8. It is a topic widely explored in the literature, where we can find, for example, texts which consider magnetic effects in the Kerr effect. For more information, see [8].

7. It is worth mentioning that there are other origins for the breaking of the linearity of the medium.

1.3 Harmonic regime

After exploring the notion of linear mediums, let us explain better the comment that was made when we wrote 2. There, we supposed that we were working in the harmonic regime, which consists, basically, of a regime in which the fields oscillate as sinusoids with fixed frequency $\omega \in [0, +\infty)$. Let us define it properly.

Definition 1.6 (Harmonic regime)**.** *Let $M \in \mathbb{R}^3$ be a medium that constitutes an electromagnetic system. Consider*

$$\vec{F}(\vec{r}, t) \in \left\{ \vec{E}(\vec{r}, t), \vec{D}(\vec{r}, t), \vec{B}(\vec{r}, t), \vec{H}(\vec{r}, t), \vec{J}(\vec{r}, t) \right\}.$$

We say that this system is in the harmonic regime if $\vec{F}(\vec{r}, t)$ can be written in the format

$$\vec{F}(\vec{r}, t) = \Re \left\{ \hat{F}(\vec{r}, \omega) e^{i\omega t} \right\},$$

for some $\hat{F} : \mathbb{R}^3 \times \{\omega\} \to \mathbb{C}$, for $\omega \in [0, +\infty)$. The notation $\hat{F}(\vec{r}, \omega)$ specifies which frequency is fixed.[9]

Even though the term harmonic is sometimes associated with the harmonic functions, i.e., eigenfunctions of the Laplace-Beltrami operator,[10] here, this adjective expresses the fact that we are using sinusoidal oscillating solutions for the fields. As sine and cosine express the solution for harmonic oscillator type equations, the terminology "*harmonic*" is inherited by the regime described in 1.6.

Pointing this out, the implementation of the harmonic regime usually reduces the complexity of the analysis and computations in electromagnetism. Given that the Maxwell equations involve the temporal derivative of the electric and magnetizing fields, in this regime, differential equations which consider terms as $\frac{\partial}{\partial t}\vec{E}(\vec{r}, t)$, for instance, can be written in a simpler way by the relation

$$\frac{\partial}{\partial t}\vec{E}(\vec{r}, t) = \Re \left\{ i\omega \hat{E}(\vec{r}, \omega)\, e^{i\omega t} \right\},$$

what is, sometimes, expressed by saying that the operator $\frac{\partial}{\partial t}$ is replaced by $i\omega$.

In addition, the harmonic regime appears naturally in the modeling of stationary states of systems that work oscillating with a fixed frequency. In other words, it is important to study long-term oscillatory phenomena, after the transient may be disregarded. Additionally, the separation between a spatial term $\hat{F}(\vec{r}, \omega)$ and a temporal exponential $e^{i\omega t}$ leads to an easier application of the method of separation of variables to solve partial differential equations concerning $\vec{F}(\vec{r}, t)$.

9. Sometimes, in the literature, the notation $\hat{F}(\vec{r})$ is used instead of $\hat{F}(\vec{r}, \omega)$ to explicitate that the regime is monofrequencial. Here, we chose to use the last one to make the frequency explicit when considering the superposition of different electromagnetic waves such that each one is monofrequencial.
10. For instance, it's common, in Physics, the use of the spherical harmonics, which are the eigenfunctions of the Laplace-Beltrami operation on the sphere $\mathbb{S}^2$.

Finally, it is worth mentioning that, for linear mediums, if $\vec{F}$ involves more than one frequency, we can use the superposition principle and the Fourier transform to analyze each component with a different frequency as independent and subjected to the harmonic regime. In order to exemplify it, consider $\omega \in [a,b] \subset [0,+\infty)$ is not fixed. Then, by linearity, we can write

$$\vec{F}(\vec{r},t) = \Re\left\{ \frac{1}{2\pi} \int_a^b \hat{F}(\vec{r},\omega) e^{i\omega t} \, \mathrm{d}\omega \right\}, \tag{13}$$

where $\hat{F}(\vec{r},\omega)e^{i\omega t}$ for each $\omega \in [a,b]$ contributes individually to the final field $\vec{F}(\vec{r},t)$. Similarly, in the case where the frequency spectrum is discrete and has cardinality n, we can rewrite 13 as

$$\vec{F}(\vec{r},t) = \Re\left\{ \sum_{k=1}^{n} \hat{F}(\vec{r},\omega_k) e^{i\omega_k t} \right\}.$$

1.4 Homogeneous and isotropic medium

Having presented the concepts of harmonic regime and linear medium, let us briefly explore the notion of homogeneity. For that, consider $M \subset \mathbb{R}^3$ a medium. Consider also the functions $\alpha : M \to \Sigma$ and $\beta : [0,+\infty) \to \Sigma$, which provide the constitutive parameters of M, i.e., Σ is the space of all non-null permittivity tensors, conductivity tensors, and magnetic permeability tensors. While α expresses the spatial dependence of these parameters, β considers their dependence on the time. For M linear,

$$\begin{aligned}
\mathrm{Im}(\alpha) &\subseteq \mathrm{Mat}_{3\times 3}\left(\mathcal{F}_{\mathbb{K}}^{+}\right) \times \mathrm{Mat}_{3\times 3}\left(\mathcal{F}_{\mathbb{K}}^{+}\right) \times \mathrm{Mat}_{3\times 3}\left(\mathcal{F}_{\mathbb{K}}^{+}\right) \\
\mathrm{Im}(\beta) &\subseteq \mathrm{Mat}_{3\times 3}\left(\mathcal{F}_{\mathbb{K}}\left(\mathbb{K}^3\right)\right) \times \mathrm{Mat}_{3\times 3}\left(\mathcal{F}_{\mathbb{K}}\left(\mathbb{K}^3\right)\right) \times \mathrm{Mat}_{3\times 3}\left(\mathcal{F}_{\mathbb{K}}\left(\mathbb{K}^3\right)\right)
\end{aligned}$$

where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and $\mathcal{F}_{\mathbb{K}}^{+}$ and $\mathcal{F}_{\mathbb{K}}\left(\mathbb{K}^3\right)$ are the spaces of functions from $[0,+\infty)$ and from $\mathbb{K}^3$, respectively, to $\mathbb{K}$. The choice between $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depends on whether the analysis considers real or complex fields, for example. A well-formulated description for a homogeneous medium may be done by the use of the group theory language and, by a better understanding of this text, let us present the idea of a group structure in algebra.

Definition 1.7 (Group)**.** *Let G be a set endowed with an operation $\circ : G \times G \to G$. $(G, \circ)$ is called a group if it has the following properties.*

1. *Associativity:* $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3), \ \forall \ g_1, g_2, g_3 \in G$*;*
2. *Existence of neutral element:* $\exists \ e \in G$ *such that* $e \circ g = g = g \circ e, \ \forall \ g \in G$*;*
3. *Existence of inverse elemente:* $\forall \ g \in G, \ \exists \ g^{-1} \in G$ *such that* $g \circ g^{-1} = e = g^{-1} \circ g$*.*

When, besides these properties, the binary operation $\circ$ is commutative, i.e. if, $g_1 \circ g_2 = g_2 \circ g_1, \ \forall \ g_1, g_2 \in G$, we say $(G, \circ)$ is an abelian group.
Moreover, let M be another set. The (left-) action of G on M is a function

$$\psi : G \times M \to M, \qquad (g,x) \mapsto \psi(g,x),$$

such that

1. *$\psi(e \circ x) = x, \; \forall \; x \in M$;*

2. *$\psi\left(g_1, \psi\left(g_2, x\right)\right) = \psi\left(g_1 \circ g_2, x\right), \; \forall \; g_1, g_2 \in G, \forall \; x \in M$.*

Usually, we just simplify the notation writing $\psi(g, x)$ as $g \circ x$. Besides that, when the action is implied, we say that G acts on M.

Group theory is a widely used area in Physics, from areas such as crystallography[11] to relativity, for instance.[12] For now, let us just illustrate it by two specific groups, where one is abelian, while the other one is not. Consider a finite $\mathbb{K}$-vector space $(V, +)$, for $\mathbb{K}$ is an arbitrary field. Then, V acts on itself by

$$\vec{x} + \vec{y} \in V, \qquad \forall \; \vec{x}, \vec{y} \in V.$$

This group is called the translation group of V. A special case of this last one is $(\mathbb{R}^n, +)$, $n \in \mathbb{N}$, with the canonical sum operation, where we write

$$\vec{x} + \vec{y} = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \in \mathbb{R}^n,$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$. It is easy to see that this group is abelian, different from the next one that we will present.

Let $\mathbb{K}$ be a field and consider $GL(n, \mathbb{K})$ the set of all invertible matrices $n \times n$ with entries in $\mathbb{K}$. Endowed with the matrix product $\cdot$, $(GL(n, \mathbb{K}), \cdot\,)$, it is a group, called the general linear group on $\mathbb{K}$. Then, consider the subset

$$O(n, \mathbb{K}) := \left\{M \in GL(n, \mathbb{K}) : M \cdot M^T = \mathbb{1} = M^T \cdot M\right\} \subset GL(n, \mathbb{K}),$$

where $\mathbb{1}$ is the identity of the linear group, which coincides with the $n \times n$ identity matrix.[13] Making $O(n, \mathbb{K})$ inherit the group operation from $GL(n, \mathbb{K})$, it becomes a subgroup of this last one, i.e., a subset of $GL(n, \mathbb{K})$ that is closed under the group operation. Let us define another subgroup, but now of $O(n, \mathbb{K})$. Define

$$SO(n, \mathbb{K}) := \{M \in O(n, \mathbb{K}) : \det M = 1\} \subset O(n, \mathbb{K}).$$

Then, $(SO(n, \mathbb{K}), \cdot\,)$ is also a group, named the special orthogonal group. Given a finite $\mathbb{K}$-vector space V of dimension n, $SO(n, \mathbb{K})$ acts on V by

$$R \cdot \vec{x} = R\vec{x}R^T$$

as a rotation.[14] For $V = \mathbb{R}^2$, this group is abelian and the order of the rotations doesn't change the final vector, however, it is not what usually happens.

Thus, having presented the notion of groups, we can express what a homogeneous medium is.

11. For an introduction to the application of group theory into topological crystallography, see this material [10]. If you want to have a more complete introduction to group theory, see [2].
12. The set S_3 of permutations which was used in 11, with the operation $\circ$ of function composition, has a group structure and $(S_3, \circ)$ is called the symmetric group of order 3. It is an important example of group, since it is the smaller non-abelian group, i.e., there is no non-abelian group with fewer elements than S_3.
13. It is not difficult to prove that the identity of a group is unique.
14. To see that, just consider a $\mathbb{K}$-inner product in V and show that $(SO(n, \mathbb{K}), \cdot\,)$ preserves it and the orientation of the space.

Definition 1.8 (Homogeneous medium and time-invariant medium)**.** *Let $(G, \circ)$ be a group that acts on M, where $\circ$ is the group operation. If, for all $g \in G$ such that $g \circ M = M$, we have*

$$\alpha(g \circ x) = \alpha(x), \qquad \forall\ x \in M,$$

we say that M is G-homogeneous. Furthermore, when G is the translation group of $\mathbb{R}^n$, M is simply called homogeneous. Similarly, if G is the translation group of $[0, +\infty)$, if, for all $g \in G$ such that $g \circ [0, +\infty) = [0, +\infty)$, holds

$$\beta(g \circ t) = \beta(t), \qquad \forall\ t \in [0, +\infty),$$

we say that M is time-invariant.

Essentially, for homogeneous mediums, $\varepsilon^{(i)}(\vec{r}, t) = \varepsilon^{(i)}(t)$ and $\sigma^{(i)}(\vec{r}, t) = \sigma^{(i)}(t)$, $\forall\ \vec{r} \in \mathbb{R}^3$, $\forall\ i \in \mathbb{N}$ or, less generally, for the relevant values of $i \in \mathbb{N}$.[15] Analogously, for time-invariant mediums, $\varepsilon^{(i)}(\vec{r}, t) = \varepsilon^{(i)}(\vec{r})$ and $\sigma^{(i)}(\vec{r}, t) = \sigma^{(i)}(\vec{r})$, $\forall\ t \in [0, +\infty)$, $\forall\ i \in \mathbb{N}$.

Intuitively, for a homogeneous medium, there is no spatial variation of the parameters that describe the electromagnetism in it, so that, roughly speaking, the last one "looks" the same at each of its points (in terms of physical measurements).

Furthermore, since homogeneity is a property of invariance under translations, it is reasonable to wonder about invariances by rotations. The reason for this reasonableness lies in the fact that all the isometries of $\mathbb{R}^n$ that preserve orientation, i.e., all the endomorphisms that preserve the canonical Euclidean product

$$\underbrace{(x_1, \ldots, x_n)}_{:=x} \cdot \underbrace{(y_1, \ldots, y_n)}_{:=y} = x_1 y_1 + \cdots + x_n y_n,\ \forall\ x, y \in \mathbb{R}^n,$$

and the orientation of the space, are translations or rotations. Guided by this principle, by observation of the equations 6 and 7, we can notice that ε_0 or, more generally, ε, and μ are considered functions with the image set being fields. For simplicity, let us refer to them as scalars, since, for each element from the domain, they are scalars. A consequence of this fact is that they do not codify directional dependencies and all directions have the same value of ε and μ. To make possible this dependence, let us define these physical quantities as follows.

Definition 1.9 (Constitutive parameters)**.** *Let M be a linear, causal and non-permanently polarized medium in harmonic regime. Let $\vec{E}(\vec{r}, t)$, $\vec{D}(\vec{r}, t) \in \mathbb{R}^3$ be an electric field and the displacement electric field, respectively, in M. The electric permittivity of M (or 1-permittivity tensor) is the function*

$$\varepsilon : \mathbb{R}^3 \times [0, +\infty) \to \operatorname{End}(\mathbb{R}^3)$$

such that, for each $t > 0$, $\varepsilon(t)$ is the $(1,1)$-tensor such that

$$\vec{D}(\vec{r}, t) = \int_0^t \varepsilon(\vec{r}, t - \tau) \vec{E}(\vec{r}, \tau)\, \mathrm{d}\tau\,.$$

15. The choice to present the definition of a homogeneous space, even after having used this notion earlier, is due to the fact that the discussion on linearity introduced the constitutive parameters gradually and followed the most common approaches found in the standard electromagnetism textbooks.

In terms of components in the frequency domain,

$$\hat{D}_i(\vec{r},\omega) = \sum_{j=1}^{3} \hat{\varepsilon}_{ij}(\vec{r},\omega) E_j(\vec{r},\omega).$$

Analogously, let $\vec{B}(\vec{r},t)$, $\vec{H}(\vec{r},t) \in \mathbb{R}^3$ *be a magnetic field and the magnetizing field, in this order, in* M. *The magnetic permeability of* M *(or* 1*-permeability tensor) is the function*

$$\mu : \mathbb{R}^3 \times [0,+\infty) \to \mathrm{End}(\mathbb{R}^3)$$

such that, for each $t > 0$, $\mu(t)$ *is the* $(1,1)$*-tensor such that*

$$\vec{B}(\vec{r},t) = \int_0^t \mu(\vec{r},t-\tau)\vec{H}(\vec{r},\tau)\,\mathrm{d}\tau\,.$$

Expressing this relation in components in the frequency domain, we obtain

$$\hat{B}_i(\vec{r},\omega) = \sum_{j=1}^{3} \hat{\mu}_{ij}(\vec{r},\omega) H_j(\vec{r},\omega).$$

Following the same principles, let $\vec{E}(\vec{r},t)$, $\vec{J}(\vec{r},t) \in \mathbb{R}^3$ *be an electric field and the conduction current field, respectively, in* M. *The conductive of* M *(or* 1*-conductive tensor) is the function*

$$\sigma : \mathbb{R}^3 \times [0,+\infty) \to \mathrm{End}(\mathbb{R}^3)$$

such that, for each $t > 0$, $\varepsilon(t)$ *is the* $(1,1)$*-tensor such that*

$$\vec{J}(\vec{r},t) = \int_0^t \sigma(\vec{r},t-\tau)\vec{E}(\vec{r},\tau)\,\mathrm{d}\tau\,.$$

In the frequency domain, component by component, we can express it as

$$\hat{J}_i(\vec{r},\omega) = \sum_{j=1}^{3} \hat{\sigma}_{ij}(\vec{r},\omega) E_j(\vec{r},\omega).$$

Based on this definition, we can define the isotropy of a medium.

Definition 1.10 (Isotropic medium)**.** *Let* M *be a linear medium.* M *is pure isotropic (or simply isotropic*[16]*) if, for all orthonormal bases,*

$$\hat{\varepsilon}(\vec{r},\omega) = \Lambda(\vec{r},\omega).\mathbb{1}, \qquad \hat{\mu}(\vec{r},\omega) = N(\vec{r},\omega).\mathbb{1}, \qquad \hat{\sigma}(\vec{r},\omega) = S(\vec{r},\omega).\mathbb{1},$$

for some $\Lambda, N, S : \mathbb{R}^3 \times [0,+\infty) \to \mathbb{C}$. *When a medium is not isotropic, we say that it is anisotropic.*

16. The term "pure" appears to distinguish it from the linear chiral mediums, also called bi-isotropic, which have coupling between the fields $\vec{E}$ and $\vec{H}$. In these cases, there are $\varepsilon, \eta, \gamma, \mu : \mathbb{R}^3 \times [0,+\infty) \to \mathrm{End}(\mathbb{R}^3)$ such that

$$\hat{D}(\vec{r},\omega) = \hat{\varepsilon}\hat{E}(\vec{r},\omega) + \hat{\eta}\hat{H}(\vec{r},\omega), \qquad \hat{B}(\vec{r},\omega) = \hat{\mu}\hat{H}(\vec{r},\omega) + \hat{\gamma}\hat{E}(\vec{r},\omega).$$

Note that this definition hides behind it an invariance already commented on: the invariance by rotations. According to the examples of groups presented before, $SO(n,\mathbb{R})$ or, more restrictly, $SO(3,\mathbb{R})$, corresponds to the group of rotations. Then, another characterization for isotropic mediums is obtained by the use of this group.

Proposition 1.1. *Let M be a linear medium with electric permittivity $\hat{\varepsilon}(\vec{r},\omega)$ and magnetic permeability $\hat{\mu}(\vec{r},\omega)$. In order to make the notation cleaner, write $\hat{\varepsilon}=\hat{\varepsilon}(\vec{r},\omega)$ and $\hat{\mu}=\hat{\mu}(\vec{r},\omega)$. Then,*

$$\hat{\varepsilon}=\Lambda(\vec{r},\omega).\mathbb{1}, \qquad \hat{\mu}=N(\vec{r},\omega).\mathbb{1}, \qquad \hat{\sigma}=S(\vec{r},\omega).\mathbb{1},$$

for some $\Lambda, N, S:\mathbb{R}^3\times[0,+\infty)\to\mathbb{C}$ if, and only if,

$$R.\hat{\varepsilon}=R\hat{\varepsilon}R^T=\hat{\varepsilon}, \quad R.\hat{\mu}=R\hat{\mu}R^T=\hat{\mu}, \quad R.\hat{\sigma}=R\hat{\sigma}R^T=\hat{\sigma}, \quad \forall\ R\in SO(3,\mathbb{R}).$$

Proof. On one hand, consider $\hat{\varepsilon}=\Lambda.\mathbb{1}$, for some $\Lambda:\mathbb{R}^3\times[0,+\infty)\to\mathbb{C}$. Then,

$$R\hat{\varepsilon}=R\hat{\varepsilon}R^T=R(\Lambda.\mathbb{1})R^T=\Lambda\left(RR^T\right)=\Lambda.\mathbb{1}=\hat{\varepsilon}$$

for all $R\in SO(3,\mathbb{R})$.
On the other hand, suppose that $R.\hat{\varepsilon}=\hat{\varepsilon}$ for all $R\in SO(3,\mathbb{R})$, especially for a rotation by $\theta\in[0,2\pi)$ around the axis z, given by

$$R_z(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$

Observe that, in fact, $R_z(\theta)\in SO(3,\mathbb{R})$, since $\det R_z(\theta)=\cos^2\theta+\sin^2\theta=1$. Moreover, write

$$\hat{\varepsilon}=\begin{pmatrix}\hat{\varepsilon}_{11} & \hat{\varepsilon}_{12} & \hat{\varepsilon}_{13}\\ \hat{\varepsilon}_{21} & \hat{\varepsilon}_{22} & \hat{\varepsilon}_{23}\\ \hat{\varepsilon}_{31} & \hat{\varepsilon}_{32} & \hat{\varepsilon}_{33}\end{pmatrix}.$$

Then, we can express $R_z(\theta)\hat{\varepsilon}R_z(\theta)^T=\hat{\varepsilon}$ explicitly as

$$\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\hat{\varepsilon}_{11} & \hat{\varepsilon}_{12} & \hat{\varepsilon}_{13}\\ \hat{\varepsilon}_{21} & \hat{\varepsilon}_{22} & \hat{\varepsilon}_{23}\\ \hat{\varepsilon}_{31} & \hat{\varepsilon}_{32} & \hat{\varepsilon}_{33}\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}\hat{\varepsilon}_{11} & \hat{\varepsilon}_{12} & \hat{\varepsilon}_{13}\\ \hat{\varepsilon}_{21} & \hat{\varepsilon}_{22} & \hat{\varepsilon}_{23}\\ \hat{\varepsilon}_{31} & \hat{\varepsilon}_{32} & \hat{\varepsilon}_{33}\end{pmatrix}$$

Computing the product of matrices above and comparing the right and the left sizes of this equation, we obtain

$$\begin{cases}\hat{\varepsilon}_{12}=\hat{\varepsilon}_{21}=0,\\ \hat{\varepsilon}_{13}=\hat{\varepsilon}_{31}=0,\\ \hat{\varepsilon}_{23}=\hat{\varepsilon}_{32}=0,\\ \hat{\varepsilon}_{11}=\hat{\varepsilon}_{22}.\end{cases}$$

As a consequence of it, $\hat{\varepsilon}$ is reduced to

$$\hat{\varepsilon} = \begin{pmatrix} \hat{\varepsilon}_{11} & 0 & 0 \\ 0 & \hat{\varepsilon}_{11} & 0 \\ 0 & 0 & \hat{\varepsilon}_{33} \end{pmatrix}$$

Similarly, consider the rotation by $\phi \in [0, 2\pi)$ around the axis x, given by

$$R_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

Then, $R_x(\phi)\hat{\varepsilon}R_x(\phi)^T = \hat{\varepsilon}$ can be written explicitly as

$$\begin{pmatrix} \hat{\varepsilon}_{33} & 0 & 0 \\ 0 & \hat{\varepsilon}_{11} & 0 \\ 0 & 0 & \hat{\varepsilon}_{11} \end{pmatrix} = \begin{pmatrix} \hat{\varepsilon}_{11} & 0 & 0 \\ 0 & \hat{\varepsilon}_{11} & 0 \\ 0 & 0 & \hat{\varepsilon}_{33} \end{pmatrix}$$

Consequently, $\hat{\varepsilon}_{11} = \hat{\varepsilon}_{33} =: \Lambda$, where $\Lambda : \mathbb{R}^3 \times [0, +\infty) \to \mathbb{C}$, and, therefore, $\hat{\varepsilon} = \Lambda.\mathbb{1}$. The demonstrations for μ and for σ follow analogously to this one.

□

Notice that, due to this proposition, the definition 1.1 could be formulated without explicitly stating that, for an isotropic medium, the electric permittivity and the magnetic permeability are proportional to the identity in all possible orthonormal bases for $\mathbb{R}^3$. If there exists one such that it happens, we can just apply a sequence of rigid movements of rotation on this basis to produce all the other ones and it will not change the matrix representation of $\hat{\varepsilon}(\vec{r}, \omega)$, $\hat{\mu}(\vec{r}, \omega)$ and $\hat{\sigma}(\vec{r}, \omega)$.

Summarily, while a homogeneous medium presents the same electromagnetic behavior independent of the point where we are analyzing within, what is usually described in the literature as "it looks the same for each one of its points", in an isotropic medium, the electromagnetic description does not depend on the direction of analysis, a property that is customarily synthesized in the expression "the medium looks the same in all directions".

Evidently, for a homogeneous and isotropic medium, it is valid $\hat{\varepsilon}(\vec{r}, \omega) = \hat{\varepsilon}(\omega).\mathbb{1}$, $\hat{\mu}(\vec{r}, \omega) = \hat{\mu}(\omega).\mathbb{1}$ and $\hat{\sigma}(\vec{r}, \omega) = \hat{\sigma}(\omega).\mathbb{1}$, where the abuse of notation for $\hat{\varepsilon}, \hat{\mu}, \hat{\sigma} : [0, +\infty) \to \mathbb{C}$ expresses the fact that we can consider the electric permittivity, the magnetic permeability and the conductivity for this medium as a function of the frequency.

1.5 The propagation of waves in a medium

After presenting some definitions that provide some characteristics of the mediums where electromagnetic waves may propagate, let us study some notions concerning this propagation.

Let M be a linear, perfect, time-invariant and non-permanently polarized medium. Then, the electromagnetism in M is governed by the Maxwell equations, given by

$$
\begin{aligned}
&\nabla \cdot \vec{D}(\vec{r},t) = \rho(\vec{r},t),\\
&\nabla \cdot \vec{B}(\vec{r},t) = 0,\\
&\nabla \times \vec{E}(\vec{r},t) = -\mu \frac{\partial}{\partial t}\vec{H}(\vec{r},t),\\
&\nabla \times \vec{H}(\vec{r},t) = \vec{J}(\vec{r},t) + \frac{\partial}{\partial t}\vec{D}(\vec{r},t),
\end{aligned}
$$

where ρ is the density of the free electric charge, which represents the charges that are not directly associated with the polarization of M and can move through the medium, while $\vec{J}$ expresses the current density of these free charges. Charges with their origin in the polarization are already considered in the definition of $\vec{D}(\vec{r},t)$ and $\vec{H}(\vec{r},t)$, given, by the linearity of M, by

$$\vec{D}(\vec{r},t) = \varepsilon(\vec{r})\vec{E}(\vec{r},t), \qquad \vec{B}(\vec{r},t) = \mu(\vec{r})\vec{H}(\vec{r},t).$$

For the majority of materials subjected to sufficiently high frequency, the magnetic effects are suppressed, especially when compared to the effects of polarization. Based on that, it is common to approximate $\mu(\vec{r}) = \mu_0$ for all $\vec{r} \in M$, particularly under the hypothesis of homogeneity, where $\mu_0 \in \mathbb{R}$ is called the vacuum magnetic permeability.[17] Having said that, supposing M homogeneous, we can rewrite the Maxwell equation considering the relations

$$\vec{D}(\vec{r},t) = \varepsilon\vec{E}(\vec{r},t), \qquad \vec{B}(\vec{r},t) = \mu_0\vec{H}(\vec{r},t).$$

Let us apply the curl on both sides of the third Maxwell equation, the Faraday law. Using the fourth equation, i.e., the Ampère-Maxwell law, and invoking Schwarz's theorem to exchange the operator curl and the time derivative, we obtain

$$\nabla \times \nabla \times \vec{E}(\vec{r},t) = -\mu_0 \frac{\partial}{\partial t}\left(\vec{J}(\vec{r},t) + \frac{\partial}{\partial t}\vec{D}(\vec{r},t)\right) = -\mu_0 \frac{\partial}{\partial t}\vec{J}(\vec{r},t) - \varepsilon\mu_0 \frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t).$$

Consequently, we can write

$$\nabla \times \nabla \times \vec{E}(\vec{r},t) + \varepsilon\mu_0 \frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t) = -\mu_0 \frac{\partial}{\partial t}\vec{J}(\vec{r},t), \tag{14}$$

what is called a non-homogeneous wave equation. In order to understand the reason for this terminology, let us use the identities for the nabla operator and rewrite the equation 14 changing the double curl by a vector Laplacian, which provides us the expression

$$\nabla^2\vec{E}(\vec{r},t) - \varepsilon\mu_0 \frac{\partial^2}{\partial t^2}\vec{E}(\vec{r},t) = \nabla\left(\nabla \cdot \vec{E}(\vec{r},t)\right) + \mu_0 \frac{\partial}{\partial t}\vec{J}(\vec{r},t). \tag{15}$$

Thus, we can observe that we obtain the already well-known wave equation, called the homogeneous wave equation, when the right side of 15 is identically null. The terms on that side are charge terms and are equal to zero if M has neither sources of charges nor free electric currents.

After having pointed it out, let us study one specific possibility of a solution for 15.

17. Even though it can be interpreted as a simple scalar, it corresponds to the tensor $\mu_0.\mathbb{1}$.

Definition 1.11 (Harmonic plane waves)**.** *An electromagnetic wave is called a harmonic plane wave if there are $\vec{k} \in \mathbb{R}^3$, $\vec{E}_0 \in \mathbb{C}^3$ and $\omega \in [0, +\infty)$ such that its electric field is given by*

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}$$

and, in an isotropic medium, $\vec{E}(\vec{r}, t), \vec{H}(\vec{r}, t)$ and $\vec{k}$ are mutually orthogonal.[18]

Finding a general solution for 15 is not a simple task; then, sometimes, it is simplified by considering only harmonic plane waves, working in the harmonic regime. Within these hypotheses, the non-homogeneous wave equation takes a very simple format, as expressed above.

Proposition 1.2. *Let M be a medium under the presented hypothesis. Consider the fields*

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}, \qquad \vec{J}(\vec{r}, t) = \vec{J}_0 e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)},$$

for $\vec{k} \in \mathbb{R}$, $\vec{E}_0, \vec{J}_0(\vec{r}) \in \mathbb{C}^3$ and $\omega \in [0, +\infty)$. Then, $\vec{E}(\vec{r}, t)$ is a solution for 15 in the harmonic regime if

$$(k^2 - \omega^2 \mu_0 \varepsilon)\vec{E}_0 = (\vec{k} \cdot \vec{E}_0)\vec{k} + i\omega \vec{J}_0.$$

for $k := \|\vec{k}\|$. In this case, $\frac{\partial^{(j)}}{\partial t^{(j)}}$, for $j \in \mathbb{N}$, acts on the fields as a product by $(-i\omega)^j$ and ∇ acts as the vector $i\vec{k}$. Moreover, when $\vec{E}(\vec{r}, t)$ is transverse and there is no free electric current, we obtain the dispersion expression

$$k^2 = \omega^2 \varepsilon \mu_0.$$

Proof. Consider $\vec{E}(\vec{r}, t)$ and $\vec{J}(\vec{r}, t)$ given according to the statement. First, as these fields are monofequencial and its physical fields are given by taking the real part of them, it is clear that we are working in the harmonic regime. Having said that, we just need to compute the elements of 15. Let us start with the spatial terms:

- $\nabla^2 \vec{E}(\vec{r}, t) = \nabla \cdot \left(\nabla \vec{E}(\vec{r}, t)\right) = -k^2 \vec{E}(\vec{r}, t);$
- $\nabla \cdot \vec{E}(\vec{r}, t) = \nabla \cdot \left[\vec{E}_0 e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}\right] = i\vec{k} \cdot \vec{E}(\vec{r}, t);$
- $\nabla \left(\nabla \cdot \vec{E}(\vec{r}, t)\right) = \nabla \left[\left(i\vec{k} \cdot \vec{E}_0\right) e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = -\left(\vec{k} \cdot \vec{E}(\vec{r}, t)\right) \vec{k}.$

Observe, by these expressions, that ∇ acts on $\vec{E}(\vec{r}, t)$ as the vector $\left(i\vec{k}\right)$, so that we can identify

$$\nabla \leftrightarrow i\vec{k}, \qquad \nabla \cdot \leftrightarrow i\vec{k}\ \cdot, \qquad \nabla \times \leftrightarrow i\vec{k}\ \times, \qquad \nabla^2 \leftrightarrow -k^2.$$

Analogously, let us calculate the terms with time derivative:

- $\frac{\partial^2}{\partial t^2} \vec{E}(\vec{r}, t) = (-i\omega)(-i\omega)\vec{E}(\vec{r}, t) = -\omega^2\ \vec{E}(\vec{r}, t);$
- $\frac{\partial}{\partial t} \vec{J}(\vec{r}, t) = -i\omega\ \vec{J}(\vec{r}, t).$

18. Note that $\vec{k}$ is not a fixed vector, but it is the parallel transportation of one vector $\vec{k}$ along the wave propagation diretion and which was, initially, at a point of position $\vec{r}$. In other words, we are saying that $\vec{E}(\vec{r}, t)$, $\vec{H}(\vec{r}, t)$ and $\vec{k}$ are tangent vectors at the point identified by $\vec{r}$, for each $t \geq 0$.

Similarly to what was noticed for the operator ∇, the derivative $\frac{\partial^{(j)}}{\partial t^{(j)}}$ acts on $\vec{E}(\vec{r},t)$ and $\vec{J}(\vec{r},t)$ by the product, for $j = 1, 2$. By the periodicity of $e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}$, we have that it can be extended to all $j \in \mathbb{N}$ and we also can identify

$$\frac{\partial^{(j)}}{\partial t^{(j)}} \leftrightarrow (-i\omega)^j, \quad \forall\, j \in \mathbb{N}.$$

Having said that, replacing the computed terms in 15, we have that

$$-k^2\vec{E}(\vec{r},t) + \omega^2\varepsilon\mu_0\vec{E}(\vec{r},t) = -\left(\vec{k}\cdot\vec{E}(\vec{r},t)\right)\vec{k} - \omega\mu_0\vec{J}(\vec{r},t),$$

or, removing the temporal dependence by dividing both sides of this equation by $e^{i\left(\vec{r}\cdot\vec{k}-\omega t\right)}$,

$$(k^2 - \omega^2\mu_0\varepsilon)\vec{E}_0 = (\vec{k}\cdot\vec{E}_0)\vec{k} + i\omega\vec{J}_0. \tag{16}$$

When the wave is transverse in the isotropic medium, $\vec{E}(\vec{r},t) \perp \vec{k}$ and, consequently, $\vec{k}\cdot\vec{E}_0 = 0$. Along with this, for a null free electric current, the equation 16, we obtain

$$(k^2 - \omega^2\mu_0\varepsilon)\vec{E}_0 = 0.$$

by the arbitrarity of $\vec{E}_0 \in \mathbb{C}^3$, we know that $k^2 - \omega^2\mu_0\varepsilon$ or, equivalently,

$$k^2 = \omega^2\mu_0\varepsilon.$$

□

Another simplification to solve equation 15 is to reduce it to homogeneous equations. For that, let us separate the analysis into the isotropic and non-isotropic cases.

Let M an isotropic medium, where a harmonic plane wave propagates. Consider an orthonormal framework $\{e_1, e_2, e_3\}$ such that e_3 is the direction of propagation. In this case, using the proposition 1.2, the equation 15 becomes

$$\left(k^2 - \omega^2\varepsilon\mu_0\right)\vec{E}_0 = 0, \tag{17}$$

where $\varepsilon, \mu_0 \in \mathbb{C}$ are considered scalars.[19] As this wave is transverse, we can define a vector $\underline{E} \in \mathbb{C}^2$ according to the identification

$$\vec{E}_0 = \begin{pmatrix} E_1 \\ E_2 \\ 0 \end{pmatrix} \longmapsto \underline{\vec{E}} := \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}.$$

This vector, which we will define more properly in the next chapter, is called the Jones vector of $\vec{E}(\vec{r},t)$. Notice that any $(E_1, E_2) \in \mathbb{C}^2$ is a solution for 17. Such symmetry, which allows us to select any point of $\mathbb{C}^2$, is called $C2$ symmetry. Additionally, since $\underline{\vec{E}}$ is not a function of the position and of the time, it is a spatial-time invariant that represents the geometry of the wave. This is one of the clues that the Jones vector may have a central role in the study of light polarization. In addition to that, we can consider the movement

19. The notation of belonging to the complex number is just to make explicity that μ_0 is taken as a scalar, despite μ_0 be a real constant.

of the electric field vector in the plane (x, y, z) for some z fixed, say $z = 0$. Due to the $C2$ symmetry, all movements of [20] $\vec{E}(\vec{r}, t)$ following its equation are possible, i.e., the field will follow the trace of one arbitrary ellipse or a segment of line.

After presenting the isotropic case, let us study the anisotropic one. Due to the break of rotation symmetry, the dependence of the electromagnetic phenomena on the directions inside of the material expresses the difficulty of dealing with, for instance, an induced polarization vector no longer parallel to the electric field. That said, let M be an anisotropic medium. In this case, ε cannot be treated simply as a scalar, as is clear by the definition 1.9. Nevertheless, ε may have a simple form, as the next result expresses.

Proposition 1.3. *Let M be a medium under the present hypothesis. Due to the conservation of energy, ε is a positive semidefinite Hermitian tensor.*[21]

Proof. Under the hypothesis of the statement, if there is neither loss of energy nor sources of energy inside of M, the volumetric density of energy is given by

$$u(\vec{r}, t) = \frac{1}{2}\Re\left\{\vec{E}(\vec{r}, t) \cdot \vec{D}(\vec{r}, t)\right\}.$$

As the energy is conserved, $\Im\left\{\vec{E}(\vec{r}, t) \cdot \vec{D}(\vec{r}, t)\right\} = 0$. Joining with the fact that the medium is linear, instantaneous and without permanent polarization, we obtain

$$\begin{aligned}
\Re\left\{\vec{E}(\vec{r}, t) \cdot \vec{D}(\vec{r}, t)\right\} &= \Re\left\{\vec{E}(\vec{r}, t) \cdot \left(\varepsilon\vec{E}(\vec{r}, t)\right)\right\} \\
&= \Re\left\{\vec{E}(\vec{r}, t)^\dagger \varepsilon \vec{E}(\vec{r}, t)\right\} \\
&= \vec{E}(\vec{r}, t)^\dagger \varepsilon \vec{E}(\vec{r}, t) \in \mathbb{R}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\vec{E}(\vec{r}, t)^\dagger \varepsilon \vec{E}(\vec{r}, t) = \overline{\vec{E}(\vec{r}, t)^\dagger \varepsilon \vec{E}(\vec{r}, t)} = \vec{E}(\vec{r}, t)^\dagger \varepsilon^\dagger \vec{E}(\vec{r}, t) \\
&\implies \vec{E}(\vec{r}, t)^\dagger \underbrace{\left(\varepsilon - \varepsilon^\dagger\right)}_{=0} \vec{E}(\vec{r}, t) = 0 \\
&\implies \varepsilon = \varepsilon^\dagger
\end{aligned}$$

Then, ε is Hermitian. Moreover, notice that, for all non-null $\vec{E}(\vec{r}, t)$ applied, $u(\vec{r}, t) \geq 0$, since it represents the energy stored in the field. So, we have that

$$\vec{E}(\vec{r}, t)^\dagger \varepsilon \vec{E}(\vec{r}, t) \geq 0.$$

Hence, ε is positive semidefinite. □

20. Sometimes, the complex and the real fields are treated as the same. When the differentiation between them is not explicit, the context will express about which one we are discussing.
21. Some authors, as [5], consider that a medium is anisotropic if, and only if, its ε, μ have strictly positive eigenvalues. It leads them to characterize an anisotropic medium as mediums that, by definition, have positive definite ε, μ. This is not our case. To see that, consider ε with two zero eigenvalues and take the framework such that ε is diagonal, with $\varepsilon_{33} \neq 0$. Consider also a plane wave with the null third entrance. Consequently, $\vec{E}^\dagger \varepsilon \vec{E} = 0$, i.e., ε is positive semidefinite.

A direct consequence of this proposition, joint to the spectral theorem, is the fact that ε has three non-negative real eigenvalues and there is a framework such that ε assumes diagonal format. When, at least, one of these eigenvalues is zero, physically, it means that the direction given by its eigenvectors does not contain any component of the electric displacement field. Since the density of energy is proportional to the product $\vec{E} \cdot \vec{D}$, the material will not store energy in this direction, having the behavior of a transparent material. Again, by the spectral theorem, we know that there is an orthonormal framework composed of eigenvectors of ε and each one determines a direction in the medium. Based on this idea, we can separate the materials, in this context, in terms of their eigenvalues, and understand the physics of these materials by the use of these directions.

Definition 1.12 (Uniaxial and biaxial materials)**.** *Let M be a medium with electric permittivity tensor ε. Consider $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in [0, +\infty)$ the eigenvalues of ε and $\{e_1, e_2, e_3\}$ the orthonormal framework formed by the eigenvectors of ε. If*

- *$\varepsilon_i = \varepsilon_j \neq \varepsilon_k$, for distinct $i, j, k = 1, 2, 3$, i.e., if ε has only two equal eigenvalues, M is called a uniaxial material;*
- *$\varepsilon_i \neq \varepsilon_j \neq \varepsilon_k \neq \varepsilon_i$, for distinct $i, j, k = 1, 2, 3$, i.e., if ε has three different eigenvalues, M is called a biaxial material.*

Moreover, the axes given by the direction of the eigenvectors of ε are called the principal axes of M.

Having said that, let us study 14 for the anisotropic case. By Gauss' law, we know that the displacement field is perpendicular to the vector $\vec{k}$. However, this perpendicularity with respect to $\vec{k}$ is not necessarily maintained for the electric field. In fact, using the linearity of M and the hypothesis that neither $\vec{D}(\vec{r}, t)$ nor $\vec{k}$ is identically null, we observe that

$$\vec{D}(\vec{r}, t) \cdot \vec{k} = 0 \Longleftrightarrow \left(\varepsilon \vec{E}(\vec{r}, t)\right) \cdot \vec{k}.$$

Consequently, if ε rotates the vector $\vec{E}(\vec{r}, t)$ in $\mathbb{R}^3$ by an angle different from multiples of π, the perpendicularity is no longer valid. Due to this fact, sometimes it is better to write the homogeneous 14 to $\vec{D}(\vec{r}, t)$ instead of $\vec{E}(\vec{r}, t)$, which produces

$$\nabla \times \nabla \times \left(\varepsilon \vec{D}(\vec{r}, t)\right) - \omega^2 \mu_0 \vec{D}(\vec{r}, t) = 0,$$

or, in the coordinates with respect to the orthonormal framework $\{e_1, e_2, e_3\}$ that we are considering,

$$\nabla \times \nabla \times \left[\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}^{-1} \begin{pmatrix} D_1(\vec{r}, t) \\ D_2(\vec{r}, t) \\ D_3(\vec{r}, t) \end{pmatrix} \right] - \omega^2 \mu_0 \begin{pmatrix} D_1(\vec{r}, t) \\ D_2(\vec{r}, t) \\ D_3(\vec{r}, t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{18}$$

By applying the condition $\vec{D}(\vec{r}, t) \perp \vec{k}$, we know that $D_3(\vec{r}, t) = 0$, since, by hypothesis, the wave is propagating along the direction given by e_3. Usually, it reduces the difficulty to solve 14. Furthermore, let us study an especiffic case, which is when each element of $\{e_1, e_2, e_3\}$ is parallel to one distinct eigenvector of ε. Then, 18 assumes the simpler form

$$k^2 \begin{pmatrix} \frac{1}{\varepsilon_1} & 0 \\ 0 & \frac{1}{\varepsilon_2} \end{pmatrix} \begin{pmatrix} D_1(\vec{r},t) \\ D_2(\vec{r},t) \end{pmatrix} - \omega^2 \mu_0 \begin{pmatrix} D_1(\vec{r},t) \\ D_2(\vec{r},t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{19}$$

Again, since $\vec{D}(\vec{r},t)$ is not identically null, the expression above is true if, and only if,

$$\begin{cases} k^2 = \varepsilon_1\, \mu_0\, \omega^2, \\ k^2 = \varepsilon_2\, \mu_0\, \omega^2. \end{cases}$$

Consequently, we can observe that, as $k > 0$, there are two possible solutions to k, say $k_1, k_2 \in [0, +\infty)$, given by

$$k_1 = \omega\sqrt{\varepsilon_1 \mu_0}, \qquad k_2 = \omega\sqrt{\varepsilon_2 \mu_0}.$$

Physically, they represent different independent modes of propagation of the wave in the medium. As $\vec{k}$ is perpendicular to $\vec{D}(\vec{r},t) = (D_1(\vec{r},t), D_2(\vec{r},t), 0)$, each non-null component of $\vec{D}(\vec{r},t)$ will travel with a $\vec{k}$ such that its norm is a solution for the above system.
In addition, observe that, for k_1, we can write

$$k_1^2 = \mu_0 \varepsilon_0 \frac{\varepsilon_1}{\varepsilon_0} \frac{\mu_0}{\mu_0} \omega^2 = \frac{\varepsilon_r\, \mu_r\, \omega^2}{c^2},$$

where $\varepsilon_r := \frac{\varepsilon_r}{\varepsilon_0}$ and $\mu_r := \frac{\mu_0}{\mu_0} = 1$. Since the refractive index associated to ε_1 is defined as $n_1 := \sqrt{\varepsilon_r \mu_r}$, we obtain

$$n_1^2 = \frac{k_1^2 c^2}{\omega^2},$$

where $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ is the light speed. Analogously, we obtain a similar expression connecting n_2 and ε_2. Hence, notice that, unlike the isotropic case, 19 does not have the C2 symmetry and the polarization may be decomposed into normal modes for the directions e_1 and e_2, i.e., components in e_1 and e_2 that propagate with refractive indexes n_1 and n_2, respectively. In other words, if the wave has components in both normal modes, the phase $\phi_i \propto n_i$, for $i = 1, 2$, of each component advances differently, leading to the phenomenon known as birefringence.

Definition 1.13 (Birefrigence)**.** *Let M be an anisotropic medium under the presented hypothesis. We say that M is birefrigent if, for at least one direction of propagation and given $\omega \in [0, +\infty)$, there are two different solutions k for*

$$k^2 \varepsilon^{-1} \vec{D}(\vec{r},t) - \omega^2 \mu_0 \vec{D}(\vec{r},t) = 0.$$

Observe that this phenomenon could be observed when we discussed the Kerr effect. Although the medium is not linear, the presence of a second electric field broke the symmetry of the medium, leading to a difference in refractive indices that influenced the x and y components of the wave electric field.

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